Microscopic Electrodynamics

Mike Reppert

September 9, 2020

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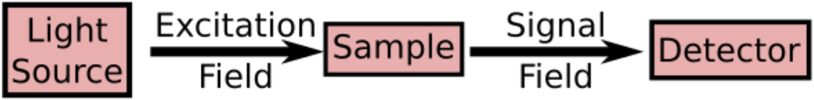
- **Lecture 3:** Examined the energy content of EM fields (via work on charged particles)
- **Lecture 4 (today):** Solve Maxwell's equations in the presence of particles (sort of...)

Why are we doing this?



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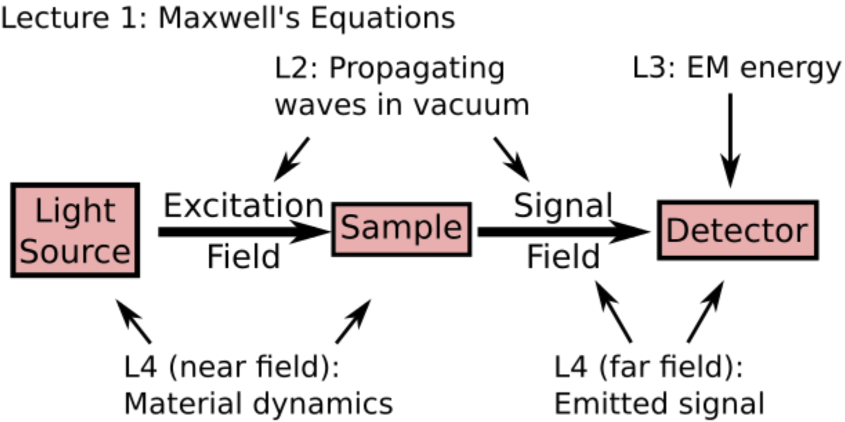

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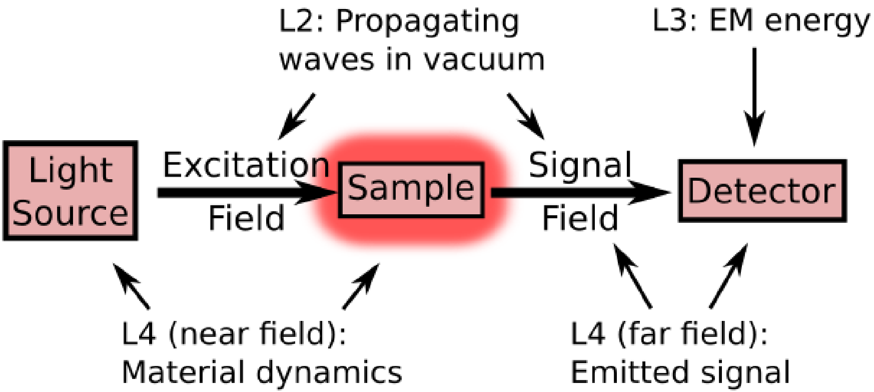
Lecture 1: Maxwell's Equations



Lecture 5: Macroscopic Maxwell's Equations
Lectures 6-9: Response Theory

Outline for Today:

1. The *In*homogeneous Wave Equation
2. The Scalar and Vector Potentials
3. Near-field vs. Far-field

The *In*homogeneous Wave Equation

Homogeneous Wave Equation

In vacuum, we rearranged Maxwell's equations

$$\begin{aligned} \nabla \cdot \boldsymbol{e} &= 0 \\ \nabla \cdot \boldsymbol{b} &= 0 \\ \nabla \times \boldsymbol{e} + \frac{1}{c}\frac{\partial \boldsymbol{b}}{\partial t} &= 0 \\ \nabla \times \boldsymbol{b} - \frac{1}{c}\frac{\partial \boldsymbol{e}}{\partial t} &= 0 \end{aligned}$$

to get the homogeneous wave equation (HWE):

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\boldsymbol{e}(\boldsymbol{x}, t) = 0.$$

*In*homogeneous Wave Equation

In the presence of charged particles

$$
\begin{aligned}
\nabla \cdot \boldsymbol{e} &= 4\pi\varrho(\boldsymbol{x}, t) \\
\nabla \cdot \boldsymbol{b} &= 0 \\
\nabla \times \boldsymbol{e} + \frac{1}{c}\frac{\partial \boldsymbol{b}}{\partial t} &= 0 \\
\nabla \times \boldsymbol{b} - \frac{1}{c}\frac{\partial \boldsymbol{e}}{\partial t} &= \frac{4\pi}{c}\boldsymbol{j}(\boldsymbol{x}, t)
\end{aligned}
$$

the same procedure yields the *in*homogeneous wave equation (IWE):

$$
\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\boldsymbol{e} = -4\pi\nabla\varrho - \frac{4\pi}{c^2}\frac{\partial \boldsymbol{j}}{\partial t}.
$$

*In*homogeneous Wave Equation

In the presence of charged particles

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Charges act as “sources” and “sinks” for the EM field!

The Inhomogeneous Wave Equation

This equation *can* be solved explicitly, **but**

1. The solutions are very complicated and
2. They *are dependent on the particle dynamics* – which are usually unknown.

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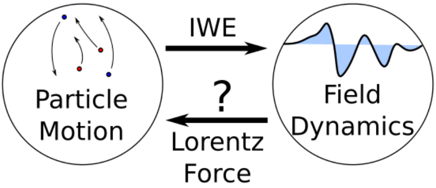
2. They *are dependent on the particle dynamics* – which are usually unknown.

Solving the IWE only gets us one way!



Take-Home Points

Maxwell's equations can be rearranged to produce the *inhomogeneous wave equation*

The IWE *can* be solved – but we need to know the particle dynamics *before* we can calculate field dynamics!

In practice, we need to approximate:

1. Assume the field is known and calculate particle dynamics or
2. Assume the particle dynamics are known and calculate the field

The Scalar and Vector Potentials

Solving the IWE

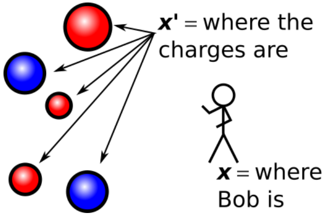
Explicit solutions to the IWE:

$$\boldsymbol{e}(\boldsymbol{x},t) = -\int d\boldsymbol{x}' \frac{\nabla' \varrho(\boldsymbol{x}',\tau)}{\|\boldsymbol{x}-\boldsymbol{x}'\|} - \frac{1}{c^2}\frac{\partial}{\partial t}\int d\boldsymbol{x}' \frac{\boldsymbol{j}(\boldsymbol{x}',\tau)}{\|\boldsymbol{x}-\boldsymbol{x}'\|}$$

$$\boldsymbol{b}(\boldsymbol{x},t) = \frac{1}{c}\int d\boldsymbol{x}' \frac{\nabla' \times \boldsymbol{j}(\boldsymbol{x}',\tau)}{\|\boldsymbol{x}-\boldsymbol{x}'\|}$$

with $\tau = t - \frac{1}{c}\|\boldsymbol{x}-\boldsymbol{x}'\|$

- $\boldsymbol{x}$ is where we observe the field
- $\boldsymbol{x}'$ runs over charge locations
- The *retarded time* τ is when the charge had to move for the signal to reach Bob at time t



https://phet.colorado.edu/sims/radiating-charge/radiating-charge_en.html

The Scalar and Vector Potentials

The solutions to the IWE can be rewritten

$$\boldsymbol{e}(\boldsymbol{x},t) = -\nabla\phi(\boldsymbol{x},t) - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$$
$$\boldsymbol{b}(\boldsymbol{x},t) = \nabla \times \boldsymbol{A}(\boldsymbol{x},t)$$

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in terms of a *scalar* potential

$$\phi(\boldsymbol{x},t) = \int d\boldsymbol{x}' \frac{\varrho(\boldsymbol{x}', t - \frac{1}{c}\|\boldsymbol{x}-\boldsymbol{x}'\|)}{\|\boldsymbol{x}-\boldsymbol{x}'\|}$$

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$$\phi(\boldsymbol{x},t) = \int d\boldsymbol{x}' \frac{\varrho(\boldsymbol{x}', t - \frac{1}{c}\|\boldsymbol{x}-\boldsymbol{x}'\|)}{\|\boldsymbol{x}-\boldsymbol{x}'\|}$$

and a *vector potential*

$$\boldsymbol{A}(\boldsymbol{x},t) = \frac{1}{c}\int d\boldsymbol{x}' \frac{\boldsymbol{j}(\boldsymbol{x}', t - \frac{1}{c}\|\boldsymbol{x}-\boldsymbol{x}'\|)}{\|\boldsymbol{x}-\boldsymbol{x}'\|}.$$

Gauge Transformations

But notice: $\boldsymbol{A}$ and ϕ are not unique! The replacement

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla f(\boldsymbol{x}, t)$$
$$\phi' = \phi - \frac{1}{c}\frac{\partial f}{\partial t}$$

leaves $\boldsymbol{e}$ and $\boldsymbol{b}$ unchanged: a *gauge transformation*.

- Our definitions so far are in the *Lorenz Gauge*.
- Also common is the *Coulomb gauge* where $\phi(\boldsymbol{x}, t)$ is just the electrostatic Coulomb potential.

Take-Home Points

Solutions to the IWE can be written as integrals over ρ and $\boldsymbol{j}$ evaluated at the *retarded time* τ and *scaled inversely by the distance* from the observer to the source charge.

These ρ and $\boldsymbol{j}$ integrals define the *scalar potential* $\phi(\boldsymbol{x}, t)$ and a *vector potential* $\boldsymbol{A}(\boldsymbol{x}, t)$.

$\boldsymbol{e}$ and $\boldsymbol{b}$ are uniquely determined by $\boldsymbol{A}$ and ϕ but not vice-versa – a *gauge transformation* changes $\boldsymbol{A}$ and ϕ but leaves $\boldsymbol{e}$ and $\boldsymbol{b}$ the same.

Near-field vs. Far-field

Approximate Solutions

In practice, solutions to the IWE are too complicated to be evaluated directly. The equations get easier in two opposite regimes:

- Near field
- Far field

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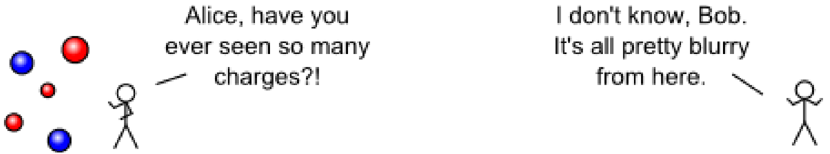


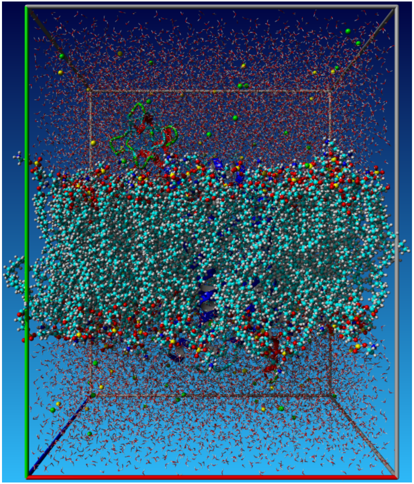
Near-field Electrodynamics

At very short distances:

- We can ignore retardation
- ϕ dominates the $\boldsymbol{e}$-field (scaling!).

$$\phi_{\mathrm{C}}(\boldsymbol{x},t) = \int d\boldsymbol{x}' \, \frac{\varrho(\boldsymbol{x}',t)}{\|\boldsymbol{x}-\boldsymbol{x}'\|} \to \sum_n \frac{q_n}{\|\boldsymbol{x}-\boldsymbol{r}_n\|}.$$

Molecular Dynamics



http://www.yasara.org/mdreport/4mbs_report.html

Far-field Electrodynamics

At very large distances:

- The retarded time is nearly the same for all sources: $\tau_r \approx t - \frac{1}{c}\|\boldsymbol{x} - \boldsymbol{x}_0\|$
- The detailed locations of the charges don't matter!

$$\phi(\boldsymbol{x}, t) \approx \frac{q_{\text{tot}}}{r} + \frac{\boldsymbol{r} \cdot \dot{\boldsymbol{\mu}}(\tau_r)}{cr^2}$$

$$\boldsymbol{A}(\boldsymbol{x}, t) \approx \frac{\dot{\boldsymbol{\mu}}(\tau_r)}{cr}$$

All determined by the total charge and **dipole moment**

$$\boldsymbol{\mu}(t) = \sum_n q_n \left(\boldsymbol{r}_n - \boldsymbol{x}_0\right)$$

relative to $\boldsymbol{x}_0$.

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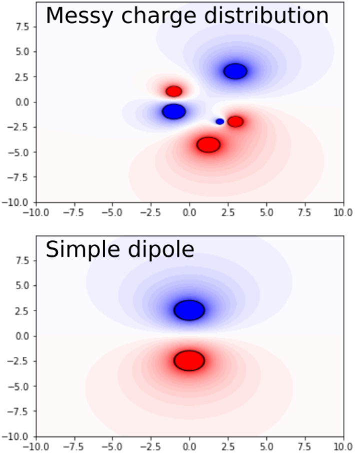
Far-field Electrodynamics

Near Field



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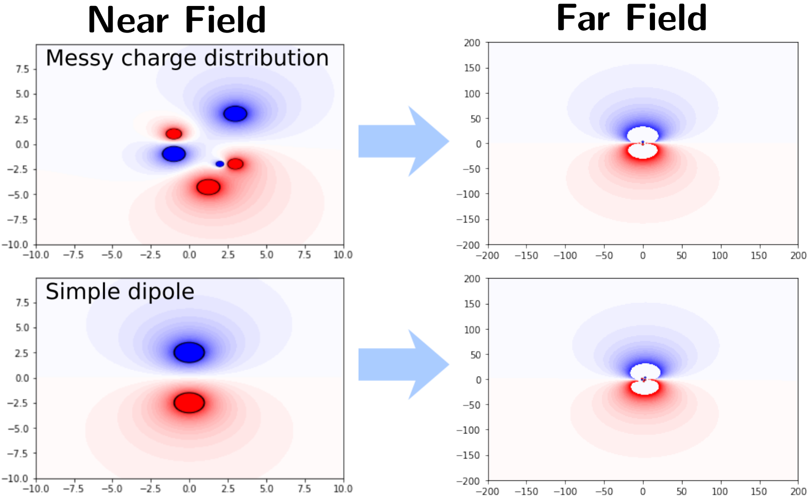
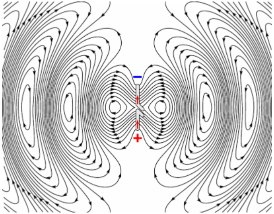


Emission of Radiation

In the far-field

- Oscillating dipoles produce propagating waves
- Everything looks like a dipole!

Oscillating Dipole:



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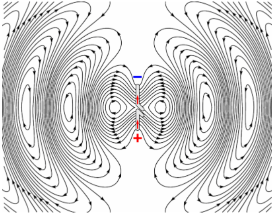

Oscillating charge distributions create propagating waves!

https://en.wikipedia.org/wiki/Antenna_(radio)

Take-Home Points

Near-field regime:

- Close to charge sources
- Coulomb potential
- Weak magnetic forces

Far-field regime:

- Far from charge sources
- Multipole expansion
- Propagating waves