Electromagnetic Waves in Vacuum

Mike Reppert

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Last time on CHEM676:

The Lorentz Force Law:

$$\boldsymbol{F}_{\mathrm{EM}} \approx q\boldsymbol{e}(\boldsymbol{r},t) + \frac{q}{c}\boldsymbol{v} \times \boldsymbol{b}(\boldsymbol{r},t)$$

Maxwell's Equations:

$$\begin{aligned}
\nabla \cdot \boldsymbol{e} &= 4\pi\varrho(\boldsymbol{x},t) \\
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Today: How does the EM field propagate in vacuum?

Outline for Today:

Decoupling the Electric and Magnetic Fields

Decoupling the E&M Fields

Maxwell's Equations in Vacuum:

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Decoupling the E&M Fields

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Decoupling the E&M Fields

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$$\Downarrow$$

$$\nabla \times (\nabla \times \boldsymbol{e}(\boldsymbol{x}, t)) + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{e}(\boldsymbol{x}, t)}{\partial t^2} = 0.$$

A Dirty Trick

Use the vector identity:

$$\nabla \times (\nabla \times \mathbf{v}) = -\nabla^2 \mathbf{v} + \nabla(\nabla \cdot \mathbf{v})$$

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or

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\boldsymbol{e}(\boldsymbol{x},t) = 0.$$

This is the **homogeneous wave equation.**

Take-Home Point

In vacuum, Maxwell's equations imply that *each component* of the electric field obeys the **homogeneous wave equation** (HWE).

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So what?

Propagating Waves

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The (one-component) wave equation

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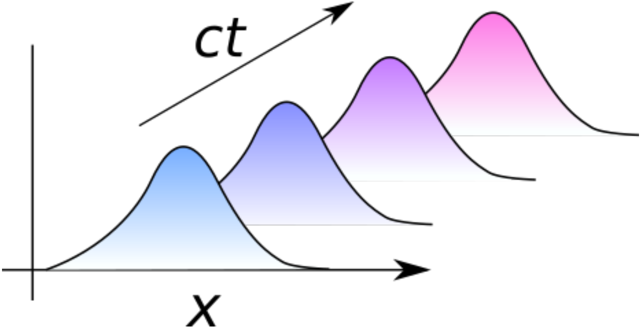
Displacement along the unit vector $\hat{\boldsymbol{s}}$ is equivalent to a shift in time, i.e. the solution *propagates* at speed c.

Take-Home Point

Solutions to the HWE can take *any form* that propagates at the speed of light.



Oscillating Signals: The Fourier Basis

The Fourier Basis

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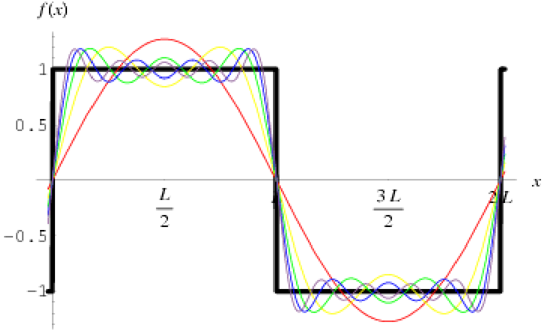
1. *Many* physical sources have well-defined frequencies
2. *All* waves can be *represented* as a *sum* of oscillatory signals



Fourier decomposition

http://mathworld.wolfram.com/FourierSeries.html

The Fourier Basis

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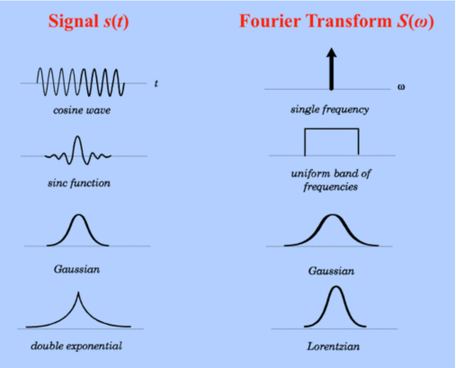

http://mriquestions.com/fourier-transform-ft.html

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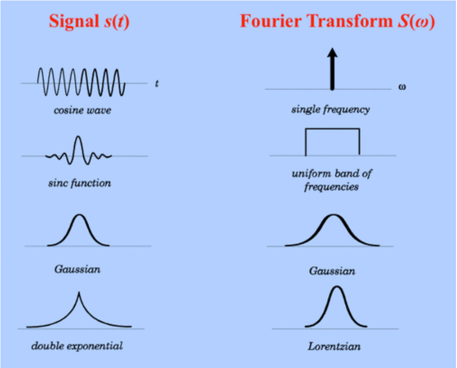


http://mriquestions.com/fourier-transform-ft.html

Note: Widths are inversely related!

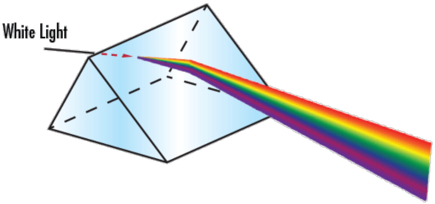
4D Fourier Transform

In electrodynamics, we use a 4D transform:

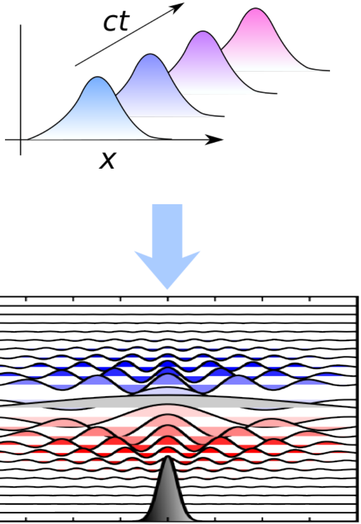
$$\tilde{\boldsymbol{e}}(\boldsymbol{k},\omega) = \int_{-\infty}^{\infty} d\boldsymbol{x} \int_{-\infty}^{\infty} dt\, \mathrm{e}^{\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} \boldsymbol{e}(\boldsymbol{x},t)$$

$$\boldsymbol{e}(\boldsymbol{x},t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\boldsymbol{k} \int_{-\infty}^{\infty} d\omega\, \mathrm{e}^{-\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} \tilde{\boldsymbol{e}}(\boldsymbol{k},\omega).$$

The individual frequency/wavevector components in $\tilde{\boldsymbol{e}}(\boldsymbol{k},\omega)$ can be physically separated using a prism!



NB: The FT is completely general! *Any field* can be decomposed as an integral of Fourier components.



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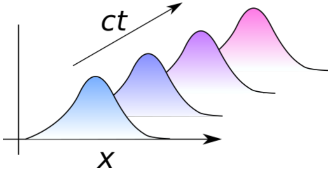

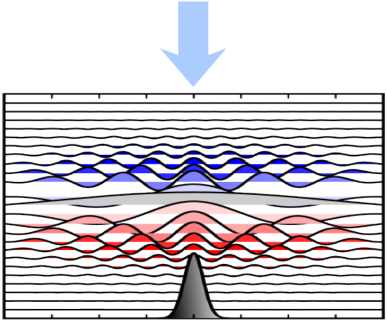

What are the characteristic features of HWE solutions in Fourier space?

The FT Derivative Property

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For the HWE, this implies

$$\left(-\frac{\omega^2}{c^2} + k^2\right) \tilde{\boldsymbol{e}}(\boldsymbol{k}, \omega) = 0$$
$$\Downarrow$$
$$\omega = ck.$$

This is the **vacuum dispersion relation** connecting frequency and wavelength $(1/k)$.

Take-Home Points

The Fourier Transform splits signals into *frequency components*.

Using a 4D FT, we can split the field into frequency components in *both time and space*.

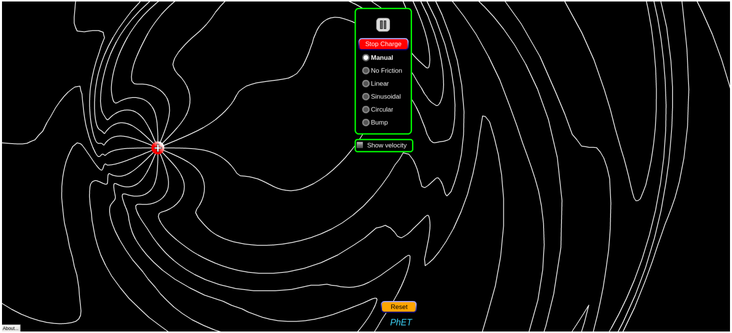
The FT converts differential equations into algebraic equations. In vacuum, the HWE implies the dispersion relation $\Rightarrow \omega = ck$.

Plane Waves

Plane Waves

In general, electromagnetic fields can be very complex!

https://phet.colorado.edu/sims/radiating-charge/radiating-charge_en.html

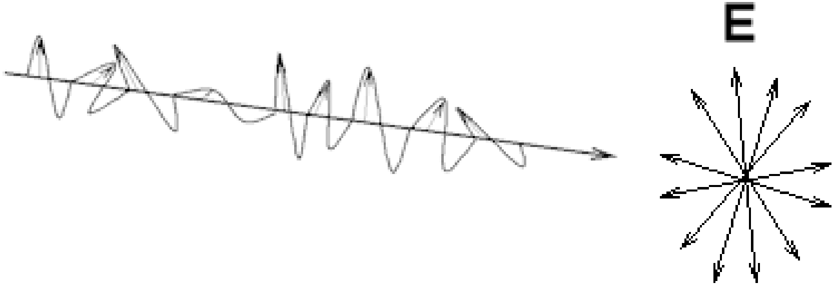


Usually, we'll consider simplified forms

$\Rightarrow$ plane waves.

Ideal Beams

A *plane wave* is an electromagnetic field propagating with a fixed $\hat{\boldsymbol{s}}$-vector.



http://labman.phys.utk.edu/phys222core/modules/m6/polarization.html

Plane Waves

General

$$\boldsymbol{e}(\boldsymbol{x},t) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d\boldsymbol{k}\int_{-\infty}^{\infty} d\omega\, \mathrm{e}^{-\mathrm{i}(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})}\tilde{\boldsymbol{e}}(\boldsymbol{k},\omega)$$

$$\Downarrow$$

Sum of Plane Waves

$$\boldsymbol{e}(\boldsymbol{x},t) = \frac{1}{2\pi}\sum_i\int_0^{\infty} d\omega\, \tilde{\boldsymbol{A}}^{(i)}(\omega)\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}\left(ct-\hat{\boldsymbol{s}}^{(i)}\cdot\boldsymbol{x}\right)} + \text{c. c.}$$

$$\Downarrow$$

Plane Wave

$$\boldsymbol{e}(\boldsymbol{x},t) = \frac{1}{2\pi}\int_0^{\infty} d\omega\, \tilde{\boldsymbol{A}}(\omega)\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}(ct-\hat{\boldsymbol{s}}\cdot\boldsymbol{x})} + \text{c. c.}$$

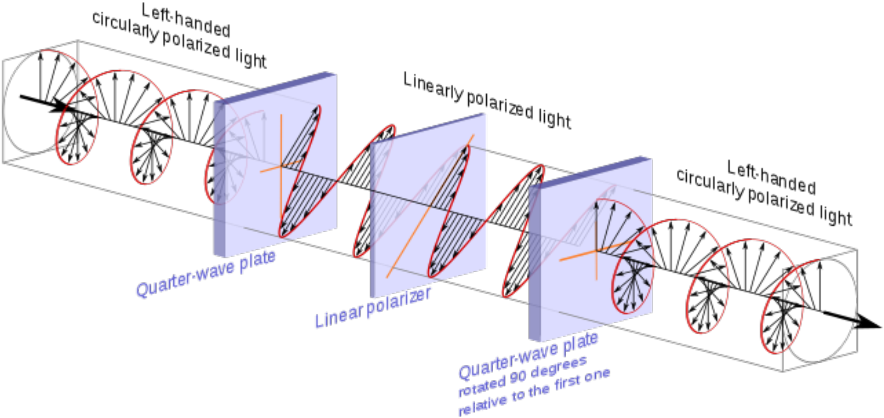
$$\Downarrow$$

Polarized Plane Wave

$$\boldsymbol{e}(\boldsymbol{x},t) = \frac{\hat{\boldsymbol{\epsilon}}}{2\pi}\int_0^{\infty} d\omega\, \tilde{A}(\omega)\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}(ct-\hat{\boldsymbol{s}}\cdot\boldsymbol{x})} + \text{c. c.}$$

Plane Waves

Polarized plane waves have both a propagation axis $\hat{s}$ *and* a polarization vector $\hat{\epsilon}$



https://en.wikipedia.org/wiki/Polarizer

Take-Home Points

In general: Electromagnetic fields are complicated!

A *plane wave* is an EM field with a well-defined propagation axis $\hat{\boldsymbol{s}}$

A *polarized plane wave* has both a propagation axis $\hat{\boldsymbol{s}}$ and a polarization vector $\hat{\boldsymbol{\epsilon}}$

Polarization comes in several flavors: Circular, eliptical, linear.